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CHARACTERIZATION OF NON-CONNECTED BUCHSBAUM CURVES IN $\mathbb{P}^n$

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In this paper we characterize non-connected Buchsbaum curves C in $\mathbb{P}^n$ and we give a sharp bound for the number of disjoint connected components of C.

Introduction.

The purpose of this note is to classify non-connected Buchsbaum curves C in $\mathbb{P}^n_k$. It is well known that the only non-connected Buchsbaum curve C in $\mathbb{P}^3_k$ is the disjoint union of two lines (cf. [4] Theorem 2.1 and [3] Remark 3.11 (5)). Moreover it is easy to check that the Hartshorne-Rao module, $M(C) := \bigoplus_{t\in\mathbb{Z}} H^1(\mathbb{P}^n, I_C(t))$, of two disjoint lines, $C = L_1 \overset{\emptyset}{\cup} L_2 \subset \mathbb{P}^3_k$, is

$$M(C)_t = \begin{cases} k & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0. \end{cases}$$

It is natural to ask whether this result generalizes to higher dimensional projective spaces and if it is possible to characterize all non-connected Buchsbaum curves $C \subset \mathbb{P}^n_k$.

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We will see that in $\mathbb{P}^4_k$ there are non-connected Buchsbaum curves $C \subset \mathbb{P}^4_k$ of arbitrary degree but, indeed, all of them have Hartshorne-Rao module

$$M(C)_t = \begin{cases} k & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0. \end{cases}$$

This result is no longer true in $\mathbb{P}^n_k, n \geq 5$. We will prove the existence of non-connected Buchsbaum curves $C \subset \mathbb{P}^n_k, n \geq 5$, of arbitrary degree with arbitrary Buchsbaum invariant (see Definition 1.3) and whose Hartshorne-Rao modules have arbitrary diameter (see Definition 1.1). Nevertheless, all of them are characterized by the following theorem:

Theorem 1. *Every non-degenerate Buchsbaum curve $C \subset \mathbb{P}^n_k$ is connected unless it is of the form $C = C_1 \cup C_2$ with C_1, C_2 disjoint Buchsbaum curves and $< C_1 > \cap < C_2 >= \emptyset$ (being $< C_i >$ the least linear subspace of $\mathbb{P}^n_k$ containing C_i).*

As application of Theorem 1, we will give, in terms of n, a sharp bound for the number of disjoint connected components of Buchsbaum curves $C \subset \mathbb{P}^n_k$ (Corollary 2.6).

In Section 1 we fix the notation and definitions needed in the sequel.

In Section 2 we prove the above theorem using algebraic tools. Then we remark the differences between the cases $n = 3, 4$ and $n \geq 5$, and we give some examples.

1. Notation and conventions.

Let k be an algebraically closed field of characteristic 0, $S = k[X_0, X_1, \ldots, X_n]$, $\mathbf{m} = (X_0, \ldots, X_n)$ and $\mathbb{P}^n =$Proj S.

By a *curve* we mean a locally Cohen-Macaulay, equidimensional, closed subscheme of $\mathbb{P}^n$ of dimension 1.

Let C be any closed subscheme of $\mathbb{P}^n$, then $I(C)$ will denote its saturated ideal and I_C its sheafification. C is said to be *degenerate* if it is contained in a hyperplane of $\mathbb{P}^n$.

We say that C has *degenerate hyperplane section* if for a general hyperplane H, $C \cap H$ is degenerate respect to H.

We will denote by $< C >$ the least linear subspace of $\mathbb{P}^n$ containing C as a subscheme.

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules we define $H^i_*(\mathcal{F}) := \bigoplus_{t \in \mathbb{Z}} H^i(\mathbb{P}^n, \mathcal{F}(t))$, $i = 0, \ldots, n$.

Definition 1.1. Given a curve $C \subset \mathbb{P}^n$, the *Hartshorne-Rao module* $M(C)$ is the graded S-module defined by

$$M(C) = \bigoplus_{t \in \mathbb{Z}} H^1(\mathbb{P}^n, I_C(t)).$$

If C is locally Cohen-Macaulay and equidimensional, then $M(C)$ has finite length and we can define the *diameter* of $M(C)$, $\operatorname{diam} M(C)$, to be the number of components from the first one different from zero to the last (inclusive).

Definition 1.2. A curve $C \subset \mathbb{P}^n$ is called *arithmetically Buchsbaum* (or simply *Buchsbaum*) if and only if $M(C)$ is annihilated by the maximal ideal $(X_0, \ldots, X_n)$ of S.

In other words, a curve C is Buchsbaum if the multiplication in $M(C)$ by any linear form is the zero map.

Definition 1.3. If $C \subset \mathbb{P}^n$ is a Buchsbaum curve, the integer

$$N = \sum_i \dim_k M(C)_i$$

is called the *Buchsbaum invariant* of C.

For instance, if $\operatorname{diam} M(C) = 1$ then C is trivially a Buchsbaum curve. This is the case of the disjoint union of two lines in $\mathbb{P}^3$. In $\mathbb{P}^3$, the simplest non-trivial example of a Buchsbaum curve is a degree 10 curve (cf. [5] Example 1.5.6). We will see in Remark 3.3 that we can find examples of Buchsbaum curves in $\mathbb{P}^7$ with $\operatorname{diam} M(C) = 2$ and $\deg C = 6$.

Remark 1.4. Let $M(C)$ be the Hartshorne-Rao module of a Buchsbaum curve $C \subseteq \mathbb{P}^n$, then $M(C)_t = 0$ for all $t < 0$. This is easy to see considering a general hyperplane H and the following exact sequence:

$$0 \longrightarrow I_C(t) \xrightarrow{\times H} I_C(t+1) \longrightarrow I_{C\cap H,H}(t+1) \longrightarrow 0 \ , \quad t \in \mathbb{Z}.$$

Taking cohomology we get the long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}^n, I_C(t)) \longrightarrow H^0(\mathbb{P}^n, I_C(t+1)) \longrightarrow \\ \longrightarrow H^0(H, I_{C\cap H,H}(t+1)) \longrightarrow M(C)_t \xrightarrow{\times H} M(C)_{t+1} \longrightarrow \cdots \end{aligned}$$

and for $t < 0$ we have

$$0 \longrightarrow M(C)_t \xrightarrow{\times H} M(C)_{t+1} \ .$$

So if C is Buchsbaum, $M(C)_t$ must be 0 for $t < 0$.

For more general results on Buchsbaum curves the reader can see, for instance, [3].

2. Non-connected Buchsbaum curves in $\mathbb{P}^n$.

We begin this section with the following basic lemma.

Lemma 2.1. *Let $C_1, C_2 \subseteq \mathbb{P}^n$ be two disjoint curves such that their union $C_1 \cup C_2$ is a non-degenerate curve, then the following two conditions are equivalent:*

(i) $< C_1 > \cap < C_2 >= \emptyset$

(ii) $I(C_1) + I(C_2) = (X_0, \dots, X_n)$

Proof. (i) $\Rightarrow$ (ii): Since $C_1 \cup C_2$ is non-degenerate $<< C_1 > \cup < C_2 >>= \mathbb{P}^n$, which together with the hypothesis (i) implies that

$$\dim < C_1 > + \dim < C_2 >= n - 1.$$

Let $r = \dim < C_1 >$, $\dim < C_2 >= n - r - 1$ and call $L_1, \dots, L_{n-r}$ the n-r k-independent linear forms in $I(C_1)$, and $H_1, \dots, H_{r+1}$ those in $I(C_2)$. They must be all k-linearly independent because $< C_1 > \cap < C_2 >= \emptyset$. Thus $\dim_k(I(C_1) + I(C_2)) = \dim_k S_1 = n + 1$, so $I(C_1) + I(C_2) = (X_0, \dots, X_n)$.

(ii) $\Rightarrow$ (i): Since $C_1 \cup C_2$ is non-degenerate, $I(C_1)$ and $I(C_2)$ cannot have any linear form in common. Therefore, ordering if necessary, we may assume under the hypothesis (ii) that $X_0, \dots, X_t \in I(C_1)$ and $X_{t+1}, \dots, X_n \in I(C_2)$, and this implies $< C_1 > \cap < C_2 >= \emptyset$. $\square$

Lemma 2.2. *Let $C = C_1 \cup C_2 \subseteq \mathbb{P}^n$ be the disjoint union of two Buchsbaum curves. Assume that C is non-degenerate and $< C_1 > \cap < C_2 >= \emptyset$, then C is a Buchsbaum curve.*

Proof. By Lemma 2.1 $< C_1 > \cap < C_2 >= \emptyset$ is equivalent to $I(C_1)+I(C_2) = (X_0, \dots, X_n)$. Consider the following exact sequence

$$0 \longrightarrow I(C) \longrightarrow I(C_1) \oplus I(C_2) \longrightarrow I(C_1) + I(C_2) \longrightarrow 0;$$

sheafifying and taking cohomology we obtain the exact diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & I(C) \arrow[r] & I(C_1) \oplus I(C_2) \arrow[rr] \arrow[dr] & & S \arrow[r] & M(C) \arrow[r] & M(C_1) \oplus M(C_2) \\
& & & \mathbf{m} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

because $H^0_*(I_{C_1} + I_{C_2}) = S$ since C_1 and C_2 are disjoint (use the Nullstellensatz). Thus, $H^0_*(I_{C_1} + I_{C_2})/(I(C_1) + I(C_2)) \cong S/\mathbf{m} \cong k$ as graded S-modules and the next sequence is exact:

$$0 \longrightarrow S/\mathbf{m} \cong k \longrightarrow M(C) \xrightarrow{\varphi} M(C_1) \oplus M(C_2) \longrightarrow 0$$

Let H be a linear form in S and consider the commutative diagram for $t \geq 1$, $t \in \mathbb{Z}$:

$$\begin{tikzcd}
M(C)_t \arrow[r, "\cong"] \arrow[d, "\times H"'] & M(C_1)_t \oplus M(C_2)_t \arrow[d, "\times H"'] \\
M(C)_{t+1} \arrow[r, "\cong"] & M(C_1)_{t+1} \oplus M(C_2)_{t+1}
\end{tikzcd}$$

Since C_1, C_2 are Buchsbaum curves, the multiplication by H in $M(C)$ must be also the zero morphism. For $t = 0$ we have

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r] & M(C)_0 \arrow[r, "\varphi_0"] \arrow[d, "\times H"'] & M(C_1)_0 \oplus M(C_2)_0 \arrow[d, "\times H"'] \\
 & 0 \arrow[r] & M(C)_1 \arrow[r, "\varphi_1"] & M(C_1)_1 \oplus M(C_2)_1
\end{tikzcd}$$

Thus, for all $f \in M(C)_0$, $\varphi_1(H.f) = H.\varphi_0(f) = 0$, and using that φ_1 is injective we conclude that C is a Buchsbaum curve. $\square$

Now we will see that the only non-connected Buchsbaum curves in $\mathbb{P}^n$ are those described in Lemma 2.2.

Theorem 2.3. *Let $C \subseteq \mathbb{P}^n$ be a non-degenerate Buchsbaum curve. Then C is connected unless $C = C_1 \cup C_2$ with C_1, C_2 disjoint Buchsbaum curves and $< C_1 > \cap < C_2 > = \emptyset$.*

Proof. By Lemma 2.1 we have to see that $I(C_1) + I(C_2) = (X_0, \ldots, X_n)$ to prove $< C_1 > \cap < C_2 > = \emptyset$.
By the following exact sequence,

$$0 \longrightarrow I(C) \longrightarrow I(C_1) \oplus I(C_2) \longrightarrow$$
$$\longrightarrow H^0_*(I_{C_1} + I_{C_2}) \longrightarrow M(C) \longrightarrow M(C_1) \oplus M(C_2) \longrightarrow \cdots$$

and since $C_1 \cap C_2 = \emptyset$, we obtain the short exact sequence:

$$0 \longrightarrow S/(I(C_1) + I(C_2)) \longrightarrow M(C) \longrightarrow M(C_1) \oplus M(C_2) \longrightarrow 0$$

Note that $k \subset S/(I(C_1) + I(C_2))$ so $k \subset M(C)_0$.
Suppose that there exists $i \in 0, \ldots, n$ such that $X_i \notin I(C_1) + I(C_2)$. Then $0 \neq [X_i] \in S/(I(C_1) + I(C_2)) \subseteq M(C)$ and the multiplication by X_i

$$M(C)_0 \xrightarrow{\times X_i} M(C)_1$$

$$1 \longmapsto X_i$$

would not be the zero map, which is in contradiction with the assumption of C being Buchsbaum.
Therefore $I(C_1) + I(C_2) = (X_0, \ldots, X_n)$ and the quotient $S/(I(C_1) + I(C_2))$ is k.

Now we have to show that C_1 and C_2 are Buchsbaum curves. We use the exact sequence of graded S-modules

$$0 \longrightarrow k \longrightarrow M(C) \xrightarrow{\varphi} M(C_1) \oplus M(C_2) \longrightarrow 0.$$

Let H be a linear form in S and consider the multiplication by H. Let $t \in \mathbb{Z}$, if $t \geq 1$ the multiplication by H in $M(C_i)_t$ is the zero map because φ_t are isomorphisms.
If $t = 0$, we have the following commutative diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & k & \longrightarrow & M(C)_0 & \xrightarrow{\varphi_0} & M(C_1)_0 \oplus M(C_2)_0 \\
 & & & \downarrow{\scriptstyle \times H} & & \downarrow{\scriptstyle \times H} \\
 & 0 & \longrightarrow & M(C)_1 & \xrightarrow{\varphi_1} & M(C_1)_1 \oplus M(C_2)_1
\end{array}$$

For all $(s_1, s_2) \in M(C_1)_0 \oplus M(C_2)_0$, there exists $f \in M(C)_0$ such that $\varphi_0(f) = (s_1, s_2)$. Now, since C is Buchsbaum, $H.f = 0$, so $\varphi_1(H.f) = 0$ and by the commutativity of the diagram we get $H.(s_1, s_2) = 0$. Thus C_1 and C_2 are also Buchsbaum curves. $\square$

Remark 2.4. Applying this theorem to the case $n = 4$, we get that the only non-connected Buchsbaum curves $C \subset \mathbb{P}^4$ are the union of a curve contained in a plane π and a line skew with π. As a consequence, every non-connected Buchsbaum curve $C \subseteq \mathbb{P}^n$, $n \leq 4$, is contained in a hyperquadric and has

$$M(C)_t = \begin{cases} k & \text{if} \quad t = 0 \\ 0 & \text{if} \quad t \neq 0. \end{cases}$$

For $n \geq 5$, as a result of Theorem 2.3, we have that every non-connected Buchsbaum curve $C = C_1 \cup C_2 \subseteq \mathbb{P}^n$ also lies in a hyperquadric Q (we can take Q equal to the union of one hyperplane containing C_1 and one containing C_2). But in the proof of the theorem we have seen that for $t \geq 1$, $M(C)_t \cong M(C_1)_t \oplus M(C_2)_t$, so we will have non-connected Buchsbaum curves for which $M(C)$ has arbitrary diameter and arbitrary Buchsbaum invariant.
In this way we can find non-connected Buchsbaum curves $C \subseteq \mathbb{P}^7$ of degree 6 and diam $M(C) > 1$:
Let $C_1 \subset H_1 \cong \mathbb{P}^3$ be the curve obtained from the union X of two disjoint lines, performing a basic double link with a plane and a quadric containing X (for the definition and facts about basic double links see, for instance, [1]). Then

$$M(C_1)_t = \begin{cases} k & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t \neq 1 \end{cases}$$

and $\deg(C_1) = 4$. Now take $C_2 \subset H_2 \cong \mathbb{P}^3$ to be the disjoint union of two lines such that $H_1 \cap H_2 = \emptyset$. If we let $C = C_1 \cup C_2 \subset \mathbb{P}^7$, then C is a Buchsbaum curve (by lemma 2.2) of degree 6 and

$$M(C)_t = \begin{cases} k^2 & \text{if} \quad t = 0 \\ k & \text{if} \quad t = 1 \\ 0 & \text{if} \quad t \neq 0, 1. \end{cases}$$

Remark 2.5. In $\mathbb{P}^3$ and $\mathbb{P}^4$ Buchsbaum non-connected curves C coincide with those non-connected curves having degenerate hyperplane section: Let $H \subseteq \mathbb{P}^n$ be a general hyperplane and consider the exact sequence

$$0 \longrightarrow I_C \xrightarrow{\times H} I_C(1) \longrightarrow I_{C\cap H,H}(1) \longrightarrow 0$$

taking cohomology we get the exact sequence

$$0 \longrightarrow H^0(I_C) \longrightarrow H^0(I_C(1)) \longrightarrow H^0(I_{C\cap H,H}(1)) \longrightarrow$$
$$\longrightarrow M(C)_0 \xrightarrow{\times H} M(C)_1 \longrightarrow \cdots.$$

C is a Buchsbaum curve, thus the last morphism is 0 and $h^0(I_{C\cap H,H}(1)) \neq 0$, i.e. the general hyperplane section of C is degenerate.

For $n = 3$, the proof results by [4]; Theorem 2.1 shows that in this case C must be the union of two skwe lines. For $n = 4$, we may use the results of [2] where curves in $\mathbb{P}^4$ are characterized.

In $\mathbb{P}^n$, $n \geq 5$ this is no longer true. Consider the following example:

In $\mathbb{P}^5$ let $C = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$, with C_1 a plane curve and C_2 the disjoint union of two lines in $\mathbb{P}^3$. Then $< C_1 > \cap < C_2 > \neq \emptyset$ (so C is not a Buchsbaum curve) and the general hyperplane section will be degenerate:

$$\dim < C \cap H > = \dim < C_1 \cap H > + \dim < C_2 \cap H > - \\ - \dim(< C_1 \cap H > \cap < C_2 \cap H >) = 1 + 1 - (-1) = 3.$$

As application of Theorem 2.3, we will bound the number of disjoint connected components of Buchsbaum curves $C \subset \mathbb{P}^n$ in terms of n and we will prove that the bound we give is optimal. To this end, for any $x \in \mathbb{R}$, set $[x] := \max\{m \in \mathbb{Z} \mid m \leq x\}$. We have

Corollary 2.6. *Let $C \subset \mathbb{P}^n$, $n \geq 2$, be a Buchsbaum curve. Denote by $m(C)$ the number of disjoint connected components of C. Then $m(C) \leq [\frac{n+1}{2}]$. Moreover there exist Buchsbaum curves $C \subset \mathbb{P}^n$ which attain this bound.*

Proof. We proceed by induction on n. Since all plane curves are connected and the only non-connected Buchsbaum curve in $\mathbb{P}^3$ is the disjoint union of two skew lines, the result is true for $n = 2$ and $n = 3$.

We assume $n \geq 4$. By Theorem 2.3 if $C \subset \mathbb{P}^n$ is a non-connected Buchsbaum curve, we can write $C = C_1 \cup C_2$ with C_1, C_2 Buchsbaum curves spanning disjoint linear subspaces. Set $< C_1 > \cong \mathbb{P}^i, < C_2 > \cong \mathbb{P}^j$ with $1 \leq i \leq j$; so $i + j \leq n - 1$. If C has the maximum number of disjoint connected components among Buchsbaum curves in $\mathbb{P}^n$, we need $i + j = n - 1$, so $j = n - 1 - i$. Now applying the induction hypothesis to the Buchsbaum curves $C_1 \subseteq \mathbb{P}^i$, and $C_2 \subseteq \mathbb{P}^{n-1-i}$, we get

$$m(C) = m(C_1) + m(C_2) \leq \left[\frac{i+1}{2}\right] + \left[\frac{n-i}{2}\right] \leq \left[\frac{n+1}{2}\right]$$

which proves what we want.

To prove the existence of curves $C \subset \mathbb{P}^n$ attaining this bound, one can consider the following curves:

(1) If n is odd, take C equal to the disjoint union of $\frac{n+1}{2}$ lines $L_1, \dots, L_{\frac{n+1}{2}}$ such that $L_t \cap < L_1 \cup \ldots \cup L_{t-1} > = \emptyset$, $t = 2, \dots, \frac{n+1}{2}$.

(2) If n is even, take C equal to the union of $\frac{n}{2} - 1$ disjoint lines $L_t, \dots, L_{\frac{n}{2}-1}$ and a curve contained in a plane π such that $L_t \cap < L_1 \cup \ldots \cup L_{t-1} > = \emptyset$, $t = 2, \dots, \frac{n}{2} - 1$, and $\pi \cap < L_1 \cup \ldots \cup L_{\frac{n}{2}-1} > = \emptyset$. These curves exist, are Buchsbaum according to Lemma 2.2, and satisfy the bound for $m(C)$. $\square$

Remark 2.7. It is easy to check by induction on n, that any curve attainig this bound for $m(C)$ is as (1),(2) in the proof of the corollary.

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